

When G^2 is a König-Egerváry graph?

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Abstract

The *square* of a graph G is the graph G^2 with the same vertex set as in G , and an edge of G^2 is joining two distinct vertices, whenever the distance between them in G is at most 2. G is a square-stable graph if it enjoys the property $\alpha(G) = \alpha(G^2)$, where $\alpha(G)$ is the size of a maximum stable set in G .

In this paper we show that G^2 is a König-Egerváry graph if and only if G is a square-stable König-Egerváry graph.

Keywords: Square of a graph; Perfect matching; Maximum stable set.

1 Introduction

All the graphs considered in this paper are finite, undirected, loopless and without multiple edges. For such a graph $G = (V, E)$ we denote its vertex set by $V = V(G)$ and its edge set by $E = E(G)$. If $X \subset V$, then $G[X]$ is the subgraph of G spanned by X . By $G - W$ we mean the subgraph $G[V - W]$, if $W \subset V(G)$.

The neighborhood of a vertex $v \in V$ is the set $N(v) = \{w : w \in V \text{ and } vw \in E\}$, and $N(A) = \cup \{N(v) : v \in A\}$, for $A \subset V$. If $|N(v)| = |\{w\}| = 1$, then v is a *leaf* and vw is a *pendant edge* of G .

By C_n , K_n , P_n we denote the chordless cycle on $n \geq 4$ vertices, the complete graph on $n \geq 1$ vertices, and respectively the chordless path on $n \geq 3$ vertices.

A stable set of maximum size will be referred as to a *stability system* of G . The *stability number* of G , denoted by $\alpha(G)$, is the cardinality of a stability system in G . Let $\Omega(G)$ denotes $\{S : S \text{ is a stability system of } G\}$.

A *matching* is a set of non-incident edges of G ; a matching of maximum cardinality $\mu(G)$ is a *maximum matching*, and a matching covering all the vertices of G is called a *perfect matching*. G is a *König-Egerváry graph* provided $\alpha(G) + \mu(G) = |V(G)|$, [1], [11].

If S is an independent set of a graph G and $H = G[V - S]$, then we write $G = S * H$. Clearly, any graph admits such representations.

Theorem 1.1 [5] *If G is a graph, then the following assertions are equivalent:*

- (i) G is a König-Egerváry graph;
- (ii) $G = S * H$, where $S \in \Omega(G)$ and $|S| \geq \mu(G) = |V(H)|$;
- (iii) $G = S * H$, where S is an independent set with $|S| \geq |V(H)|$ and $(S, V(H))$ contains a matching M of size $|V(H)|$.

G is *well-covered* if it has no isolated vertices and if every maximal stable set of G is also a maximum stable set, i.e., it is in $\Omega(G)$ [8]. G is called *very well-covered* [2], provided G is well-covered and $|V(G)| = 2\alpha(G)$. Some interrelations between well-covered and König-Egerváry graphs were studied in [3], [4].

The distance between two vertices $v, w \in V(G)$ is denoted by $\text{dist}_G(v, w)$, or $\text{dist}(v, w)$ if no ambiguity. G^2 denotes the second power of graph G , i.e., the graph with the same vertex set V and an edge is joining distinct vertices $v, w \in V$ whenever $\text{dist}_G(v, w) \leq 2$. Clearly, any stable set of G^2 is stable in G , as well, while the converse is not generally true. Therefore, we may assert that $1 \leq \alpha(G^2) \leq \alpha(G)$. Let notice that the both bounds are sharp. For instance, if:

- G is not a complete graph and $\text{dist}(a, b) \leq 2$ holds for any $a, b \in V(G)$, then $\alpha(G) \geq 2 > 1 = \alpha(G^2)$; e.g., for the n -star graph $G = K_{1,n}$, with $n \geq 2$, we have $\alpha(G) = n > \alpha(G^2) = 1$;
- $G = P_4$, then $\alpha(G) = \alpha(G^2) = 2$.

The graphs G for which the upper bound of the above inequality is achieved, i.e., $\alpha(G) = \alpha(G^2)$, are called *square-stable*; e.g., the graph from Figure 1.

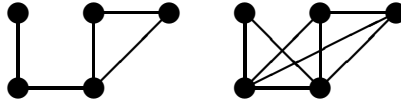


Figure 1: A square-stable graph G and its G^2 .

Theorem 1.2 [6] *The graph G is square-stable if and only if there is some $S \in \Omega(G)$ such that $\text{dist}_G(a, b) \geq 3$ holds for all distinct $a, b \in S$.*

In this paper we prove that G^2 is a König-Egerváry graph if and only if G is a square-stable König-Egerváry graph. In particular, we deduce that the square of the tree T is a König-Egerváry graph if and only if T is well-covered.

2 Results

It is quite evident that G and G^2 are simultaneously connected or disconnected. Thus in the rest of the paper all the graphs are connected.

Lemma 2.1 *If G is a square-stable graph with 2 vertices at least, then $\alpha(G) \leq \mu(G)$.*

Proof. According to Theorem 1.2 there exists a maximum stable set

$$S = \{v_i : 1 \leq i \leq \alpha(G)\}$$

in G such that $\text{dist}_G(a, b) \geq 3$ for all pairwise distinct $a, b \in S$. It follows that for every $i \in \{1, 2, \dots, \alpha(G) - 1\}$ there is a shortest path in G , of length 3 at least, connecting v_i to $v_{\alpha(G)}$, say $v_i, w_i, \dots, w^i, v_{\alpha(G)}$ (see Figure 2).

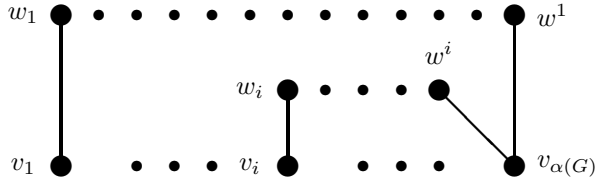


Figure 2: $S = \{v_1, \dots, v_i, \dots, v_{\alpha(G)}\} \in \Omega(G)$ and $M = \{v_1 w_1, \dots, v_i w_i, \dots, v_{\alpha(G)} w^1\}$ is a matching in G .

All the vertices $w_i, 1 \leq i \leq \alpha(G) - 1$ and w^1 are pairwise distinct, i.e.,

$$w_i \neq w^1, 1 \leq i \leq \alpha(G) - 1,$$

because, otherwise, there will be a pair of vertices in S at distance 2, in contradiction with the hypothesis on S . Hence we deduce that

$$M = \{v_i w_i : 1 \leq i \leq \alpha(G) - 1\} \cup \{v_{\alpha(G)} w^1\}$$

is a matching in G that saturates all the vertices of $S \in \Omega(G)$. Consequently, we obtain $\alpha(G) = |S| = |M| \leq \mu(G)$. ■

Remark 2.2 *The vertex w^1 in the proof of Lemma 2.1 may be a common vertex for more shortest paths connecting various v_i to $v_{\alpha(G)}$ (see Figure 3).*

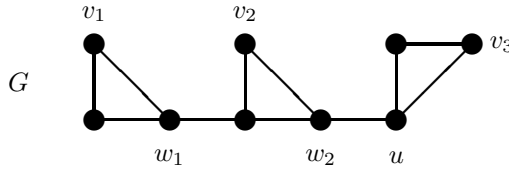


Figure 3: G has $\alpha(G) = \alpha(G^2) = 3 = |\{v_1 w_1, v_2 w_2, v_3 u\}| < \mu(G)$, where $w^1 = w^2 = u$.

The graph G in Figure 1 is square-stable and has $\mu(G) = \mu(G^2) = 2$, while the square-stable graph G from Figure 4 satisfies $\mu(G) < \mu(G^2)$. Notice that, in the both examples, neither G nor G^2 is a König-Egerváry graph.

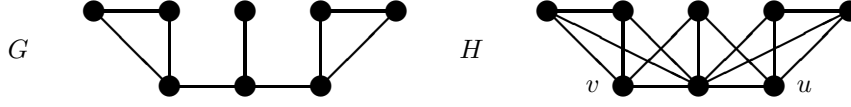


Figure 4: $G^2 = H + vu$ has $\alpha(G^2) = \alpha(G)$, while $\mu(G) < \mu(G^2)$.

Proposition 2.3 *Let G^2 be a König-Egerváry graph with 2 vertices at least. Then the following assertions are equivalent:*

- (i) $\alpha(G) = \alpha(G^2)$;
- (ii) $\mu(G) = \mu(G^2)$;
- (iii) G is a König-Egerváry graph with a perfect matching.

Proof. The following inequalities are true for every graph G :

$$\mu(G) \leq \mu(G^2) \text{ and } \alpha(G^2) \leq \alpha(G).$$

Since G^2 is a König-Egerváry graph, $\mu(G^2) \leq \alpha(G^2)$. Consequently, we get

$$\mu(G) \leq \mu(G^2) \leq \alpha(G^2) \leq \alpha(G).$$

(i) \implies (ii),(iii) If G is square-stable, then these inequalities together with Lemma 2.1 give

$$\mu(G) = \mu(G^2) = \alpha(G^2) = \alpha(G).$$

Moreover, we infer that

$$|V(G)| = \mu(G^2) + \alpha(G^2) = \mu(G) + \alpha(G),$$

which means that G is a König-Egerváry graph. In addition, G has a perfect matching, because $\mu(G) = \alpha(G)$.

(iii) \implies (i) If G is a König-Egerváry graph with a perfect matching, then

$$\mu(G) + \alpha(G) = |V(G)| = \mu(G^2) + \alpha(G^2) \text{ and } \mu(G) = \mu(G^2).$$

Thus, we deduce that $\alpha(G) = \alpha(G^2)$, i.e., G is a square-stable graph.

(ii) \implies (i) If $\mu(G) = \mu(G^2)$, then it follows that

$$|V(G)| = \alpha(G^2) + \mu(G^2) \leq \alpha(G) + \mu(G^2) = \alpha(G) + \mu(G) \leq |V(G)|,$$

which assures that $\alpha(G) = \alpha(G^2)$, i.e., G is a square-stable graph. ■

It is worth noticing that if G is square-stable, then it is not enough to know that $\mu(G) = \alpha(G)$ in order to be sure that G is a König-Egerváry graph; e.g., the graph from Figure 1.

Remark 2.4 *There are König-Egerváry graphs, whose squares are not König-Egerváry graphs; e.g., every even chordless cycle.*

Remark 2.5 *There are non-König-Egerváry graphs, whose squares are not König-Egerváry graphs; e.g., every odd chordless cycle.*

Theorem 2.6 *If G^2 is a König-Egerváry graph, then G is a square-stable König-Egerváry graph with a perfect matching.*

Proof. Since G^2 is a König-Egerváry graph, Theorem 1.1 ensures that $G^2 = S * H$, where $S \in \Omega(G^2)$, $\mu(G^2) = |V(H)|$ and every maximum matching of G^2 is contained in $(S, V(H))$.

Let $S = \{s_j : 1 \leq j \leq \alpha(G^2)\} \in \Omega(G^2)$ and $V(H) = \{h_k : 1 \leq k \leq |V(G)| - \alpha(G^2)\}$.

Claim 1. Every $h \in V(H)$ is joined, by an edge from G , to at most one vertex of S .

Otherwise, if some $h \in V(H)$ has two neighbors $s_i, s_j \in S$ such that $hs_i, hs_j \in E(G)$, then $s_i s_j \in E(G^2)$, in contradiction to the fact that S is independent.

Claim 2. $S_G(H) = S_{G^2}(H)$, where

$$S_G(H) = \{s \in S : (\exists) hs \in E(G), h \in V(H)\}, \text{ and} \\ S_{G^2}(H) = \{s \in S : (\exists) hs \in E(G^2), h \in V(H)\}.$$

Since $E(G) \subseteq E(G^2)$, we get that $S_G(H) \subseteq S_{G^2}(H)$. Assume that there is some $s \in S_{G^2}(H) - S_G(H)$. Hence, it follows that there is some $h_j s \in E(G^2) - E(G)$. Consequently, in G must exist some path on two edges from s to h_j , and because S is stable, it follows that there is some $h_k \in V(H)$, such that $h_k h_j, h_k s \in E(G)$ and this contradicts the fact that $s \in S_{G^2}(H) - S_G(H)$.

Claim 3. There is a maximum matching in G^2 containing only edges from G .

Combining *Claim 1* and *Claim 2*, it follows that every $h \in V(H)$ is joined, by an edge from G , to exactly one vertex of S , say $s(h)$, because, otherwise, we get $S_G(H) \neq S_{G^2}(H)$. Now, the set $M = \{hs(h) : h \in V(H)\}$ is a matching both in G and in G^2 . Moreover, by Theorem 1.1, M is a maximum matching in G^2 , because $|M| = |V(H)|$. Consequently, we deduce that $|M| \leq \mu(G) \leq \mu(G^2) = |M|$, which implies $\mu(G) = \mu(G^2)$.

According to Proposition 2.3, it follows that G is a square stable König-Egerváry graph having a perfect matching. ■

Notice that the converse of Theorem 2.6 is not generally true; e.g., $G = C_{2n}, n \geq 2$.

Now we are ready to formulate the main finding of the paper.

Theorem 2.7 *For a graph G of order $n \geq 2$ the following assertions are equivalent:*

- (i) G^2 is a König-Egerváry graph;
- (ii) G is a square-stable König-Egerváry graph;
- (iii) G has a perfect matching consisting of pendant edges;
- (iv) G is very well-covered with exactly $\alpha(G)$ leaves.

Proof. The implication (i) \implies (ii) follows from Theorem 2.6. The proof of the implication (ii) \implies (i) is in the following series of inequalities:

$$|V(G)| = \alpha(G) + \mu(G) = \alpha(G^2) + \mu(G) \leq \alpha(G^2) + \mu(G^2) \leq |V(G^2)| = |V(G)|.$$

All the equivalences between (ii), (iii) and (iv) have been proved in [7]. ■

It was shown in [10] that a tree having at least two vertices is well-covered if and only if it has a perfect matching consisting of pendant edges. It was also mentioned there that every well-covered tree of order at least two is very well-covered as well. Combining these observations with Theorem 2.7 we obtain the following.

Corollary 2.8 *The square of a tree is a König-Egerváry graph if and only if the tree is well-covered.*

3 Conclusions

Recall that $\theta(G)$ is the *clique covering number* of G , i.e., the minimum number of cliques whose union covers $V(G)$; $i(G) = \min\{|S| : S \text{ is a maximal stable set in } G\}$, and $\gamma(G) = \min\{|D| : D \text{ is a minimal domination set in } G\}$. In general, it can be shown that the graph invariants mentioned above are related by the following inequalities:

$$\alpha(G^2) \leq \theta(G^2) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \theta(G),$$

which turn out to be equalities, when $\alpha(G^2) = \alpha(G)$ or $\theta(G^2) = \theta(G)$ [9].

It seems interesting to find out some other graph operations and invariants such that interrelations between them may lead to König-Egerváry graphs.

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